GENERAL RELATIVITY

THE SCHWARZSCHILD SOLUTION APPLIED TO MERCURY'S ORBIT

Frigola Manzano, Oriol

1-INTRODUCTION

Einstein's theory of General Relativity tries to explain what gravity is, how it is produced, how it behaves and what effects it has. Another theory that had the same purpose was the law of universal gravitation that Newton postulated in 1686. In fact, Einstein's Theory, applied to situations where gravitational fields are weak and the velocities of the falling objects are not relativistic (much smaller than the speed of light), reproduces Newton's Law. This elucidates the limitations of Newtonian Theory and at the same time demonstrates that the two theories perfectly agree with each other in regimes where they can both be applied. However, the two theories give very different explanations of what gravity and space-time conceptually are. Newton's approach gives great accuracy in calculations and only the most modern technology may notice the error; due to that general relativity was not noticed and demonstrated[1] experimentally until the late twentieth century. Two of the most known tests are the light deflection observation from stars near the Sun and the precession of the perihelion of Mercury, which will be discussed in this paper.

The theory of General Relativity was published by Albert Einstein in 1916, i.e, eleven years after the theory of Special Relativity, which was also proposed by him. Einstein may not had discovered the second without the first theory, since there is a close relationship between them; in fact, special relativity is a special case of the general one (hence the name).

The main objective of this paper is to explain the implications of one of the solutions of GR[2], the so-called Schwarzschild metric or solution, in honor to Karl Schwarzschild, the scientist who discovered it only a few months after the publication of Einstein's theory.

Addressing the main issue, part 5, implies discussing the key points of General Relativity, ie, giving an idea of the main postulates and equations. After that, the Schwarzschild metric will be presented and finally the effects expected on the motion of Mercury will be examined. In the next section it will be discussed how Einstein interpreted the concept of gravity.

---

[1] Actually impossible to prove a theory experimentally as it would require endless experiments. Theories also have to be tested to verify its effectiveness and that was a big challenge for the scientific community during the twentieth century. A good book that explains all the experiments related to General Relativity is Will (1986) - *Was Einstein Right?*-.

[2] GR means General Relativity as SR means Special Relativity.

2. GRAVITY AS CURVATURE

G.R. starts from an idea called equivalence principle, which states that there is no difference between being floating in space, free of gravity, and under the effect of a gravitational field but in free-fall. This principle also says that remaining on the surface of the earth is the same as being subjected to a constant acceleration motion, as it might be inside a rocket. Therefore, the objects that are placed on earth would actually be the non inertial, and free falling would be the inertial ones. This means that the particle’s natural state (when there are no forces acting on it) is to fall freely. Another key point is that all bodies fall equally regardless of their mass. We know that if a force is applied to an object and measured the resulting acceleration, its mass can be obtained by Newton's second law. Also, if we measured the weight of the same object with a balance and dividing it by the gravity *g*, we will obtain the mass measured at the beginning. Actually it is not evident that this assumption must be correct, scientists did a lot of tests that showed, with great accuracy, that it was true. This suggests that gravity has less to do with “bodies” rather to a property of space surrounding the "source”.

The main objectives of the GR are to give an explanation that does not imply the action of any force as gravity, that the effect doesn’t dependent on the composition or mass of the body and to show that the natural state is free-falling; all of this being consistent with the laws the S.R.

Trying to explain all these concepts is only possible if we consider time in the same coordinate system as in special relativity *(t, x, y, z)*. Although we consider that we are static when we haven’t a spatial velocity, we are moving at high speed over time.

Fig. 1 shows that bodies that move freely through the space-time seem to fall. In Fig. 1.a the object O’ is going only in the time direction (following the red line). In Fig. 1.b the object O has a significant mass and O’ is falling in but following also a ‘straight line’. The effect is explained by the reason that the two temporal axes (the respective observers) are not parallel due to the space-time deformation caused by the presence of a mass. Formally said - *inertial systems follow a geodesic of space-time unless a force acts on them* -. This is a key point of Einstein’s explanation of what gravity is. As a clarification, a geodesic is the shortest line between two points on a curved surface or volume. Mathematically it is represented as a curve that parallel transports its tangent vector.

2.1 Covariant Derivatives

How can one infer if a vector changes or moves through parallel coordinates if they are "deformed"? The answer is keeping in mind that the basis vectors (vectors that set the coordinate axes) change along one direction, so its derivative isn’t zero.

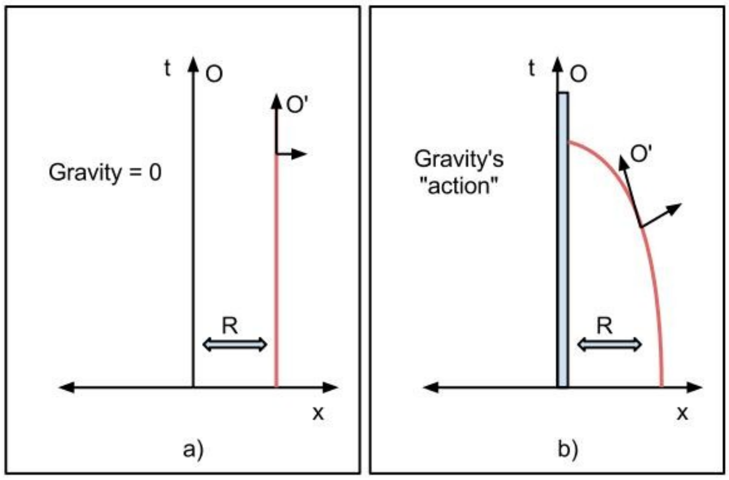

Figure 1.- In a) the particle O' moves in the same way as O, so the distance R remains constant. However, in b), O has a significant mass and O' takes also a straight trajectory but now only is straight in its own reference system! The distance R is not constant because the time direction for O' is not the same as for O. Space-time is warped.

If space is deformed and base vectors change, it may happen that two vectors have the same direction but different components simply because they are at different points of a reference system. The goal now is to describe mathematically the trajectory of a free-falling body assuming that it follows a geodesic. First, we define the derivative of a vector in which to consider the basis vectors vary and consequently must be derived. This concept of non-constancy appears in Eq. 3 where axes and vectors carry sub/superscripts *(Alpha, Beta, Gamma).* Henceforth this labeling indicates the various components of the element *(t, x, y, z)* it can also be specified with numbers from 0 to 3 with *0 = t, 1 = x, 2 = y, 3* =z.

$$\vec{V} = V^{\alpha}\vec{e}_{\alpha} \rightarrow \frac{\partial \vec{V}}{\partial x^{\beta}} = \frac{\partial}{\partial x^{\beta}} \cdot (V^{\alpha} \cdot \vec{e}_{\alpha}) = \frac{\partial V^{\alpha}}{\partial x^{\beta}} \cdot \vec{e}_{\alpha} + V^{\alpha} \cdot \frac{\partial \vec{e}_{\alpha}}{\partial x^{\beta}} \tag{3}$$

With *V* as a vector to be derived, *e* as the basis vectors, *x* as the coordinate axes.

It is important to remember that throughout this paper will use the Einstein summation convention, where if two terms have equal indexes but one as a superscript and other as a subscript, then it's understood a summation over them, as shown below:

$$V^{\alpha}\vec{e}_{\alpha} = \sum_{\alpha} V^{\alpha}\vec{e}_{\alpha} \tag{4}$$

Eq.3 is very important in differential geometry, in curved space the basis vectors change at every point. To simplify are used symbols that carry the name of its inventor Elwin Christoffel, mathematician and physicist of the nineteenth century. The *Christoffel symbols* are defined as the partial derivative with respect to the axis of the base vectors (Eq. 5),

$$\frac{\partial \vec{e}_{\alpha}}{\partial x^{\beta}} = \Gamma^{\gamma}_{\alpha\beta}\ \vec{e}_{\gamma} \tag{5}$$

The new symbol has three index vector indicating the basis to derive (Alpha) the axis on which derives (Beta) and the axis that represents the variation (range) and that it is only value or a component of the vector base. Thus the derivative with respect to the axis of the vector V, would be simplified as follows:

$$\frac{\partial \vec{V}}{\partial x^{\beta}} = \frac{\partial V^{\alpha}}{\partial x^{\beta}} \cdot \vec{e}_{\alpha} + V^{\alpha} \cdot \Gamma^{\gamma}_{\alpha\beta} \ \vec{e}_{\gamma} \tag{6}$$

Interchanging index gamma and alpha from the last two terms of Eq. 6 we get:

$$\frac{\partial \vec{V}}{\partial x^{\beta}} = \frac{\partial V^{\alpha}}{\partial x^{\beta}} \cdot \vec{e}_{\alpha} + V^{\gamma} \cdot \Gamma^{\alpha}_{\gamma\beta} \ \vec{e}_{\alpha} \tag{7}$$

$$\frac{\partial \vec{V}}{\partial x^{\beta}} = \left(\frac{\partial V^{\alpha}}{\partial x^{\beta}} + V^{\gamma} \cdot \Gamma^{\alpha}_{\gamma\beta} \right) \vec{e}_{\alpha} \tag{8}$$

The part in brackets in Eq. 8 is called *covariant derivative* and is defined as:

$$\frac{\partial V^{\alpha}}{\partial x^{\beta}} + V^{\gamma} \cdot \Gamma^{\alpha}_{\gamma\beta} \equiv \bigtriangledown \vec{V} \tag{9}$$

The geodesic curve can be defined as the one that parallel-transports its own tangent vector U. Therefore, the covariant derivative of the vector along the geodesic is zero. This means that the components of the vector changes only due to the change in basis vectors.

Next step is to derive the vector U along the parametrization determined with lambda. The tangent vector is defined as:

$$U^{\alpha} = \frac{\partial x^{\alpha}}{\partial \lambda} \tag{10}$$

If the premise is that the covariant derivative with respect to the parameter (lambda) of the tangent vector must be zero because it is a geodesic, we find that:

$$\vec{U} \cdot \bigtriangledown \vec{U} \equiv \bigtriangledown_{\vec{U}} \vec{U} = \frac{\partial x^{\beta}}{\partial \lambda} \cdot \frac{\partial U^{\alpha}}{\partial x^{\beta}} + \frac{\partial x^{\beta}}{\partial \lambda} \cdot U^{\gamma} \cdot \Gamma^{\alpha}_{\gamma\beta} = 0 \tag{11}$$

Using the definition of tangent vector and Eq. 11 we have:

$$\bigtriangledown_{\vec{U}} \vec{U} = \frac{\partial^2 x^{\alpha}}{\partial \lambda^2} + \frac{\partial x^{\beta}}{\partial \lambda} \cdot \frac{\partial x^{\gamma}}{\partial \lambda} \cdot \Gamma^{\alpha}_{\gamma\beta} = 0 \tag{12}$$

This is the geodesic equation and is of great importance since it explains how do bodies move under the action of gravity. However the application of this equation will be left for later on.

Actually there are other ways to derive the geodesic equation, such as founding it by determining that it is also the shortest trajectory between points of any given space-time.

2.2 Parallel transport

It was previously shown that the geodesic curve parallel transports its tangent vector. This transport displayed graphically in a sphere as it has a curved surface (a test that would be that it's not possible to cover with paper a ball without getting wrinkles). If you complete a triangle on the surface of a sphere you will notice that the interior angles are greater than 180 degrees and if you transport a vector along the three sides and return at the start you will detect some deviation from the original vector (Fig. 2):

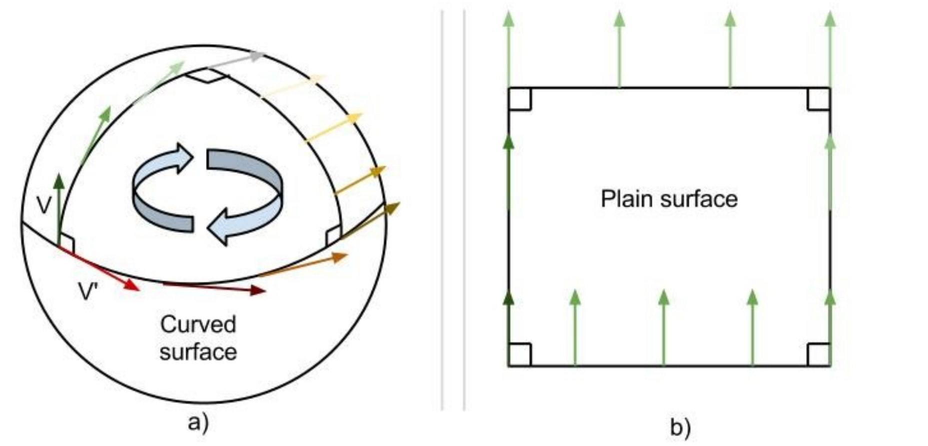

Figure 2.- In a), because the surface of a sphere is curved, a circular path produces a deviation to any parallel transported vector like V. But this doesn't happen in a plain surface.

This deviation can be calculated and depends on the distance along the inner area and the type of curvature. From that can be found a tensor that relates these concepts (area, basis vectors, Christoffel symbols, etc.) and gives the variation on the initial vector. The calculation of this tensor requires a long series of messy calculus so it’s not shown here, on the other hand, we will discuss the consequences of this and other tensors that will appear hereafter. This function is called curvature tensor or the Riemann curvature tensor in honor to the nineteenth century mathematician known for his contributions to differential geometry.

Riemann tensor has the following form and is gotten from the Christoffel symbols:

$$R^{\gamma}_{\alpha\mu\beta} \equiv \Gamma^{\gamma}_{\alpha\beta,\mu} - \Gamma^{\gamma}_{\alpha\mu,\beta} + \Gamma^{\gamma}_{\eta\mu}\Gamma^{\eta}_{\alpha\beta} - \Gamma^{\gamma}_{\eta\beta}\Gamma^{\eta}_{\alpha\mu} \qquad (13)$$

In the Eq. 13 appear some "," at the right of the first two terms, and they mean, hereinafter, left term’s derivative about the axis with the exponent that is indicated at the right of the comma. This tensor is very important because it shows the curvature in a more direct way, therefore, in a flat space R is zero.

The reason that the tensor depends on the Christoffel symbols and these on basis vectors, prove that one can know if he is in a curved space-time by doing experiments as transport parallel Fig. (2) or simply by calculating the Riemann curvature tensor. Everything can be done noticed from the "inside" of the space so you don’t have to leave it.

Now that we have a curvature tensor, it wouldn’t be strange to think that if the theory of General Relativity tries to explain gravity as a curvature of space-time, the final formula should be something like “curvature = constant ·mass-energy” because the curvature can be caused by the mass and/or energy. The Riemann tensor expresses the curvature but there is too much “information” in it because to express the "mass-energy" it’s needed only a tensor of order two (with two index) and the tensor R has four. There is a mathematical operation called index contraction that implies a summation over some components (giving values to some general indexes of the tensor) made in Eq. 14:

$$R_{ij} = R^k_{ij\,k} = R^0_{ij0} + ... + R^3_{ij3} \tag{14}$$

A contraction can be made on any other possibilities but all canceled at the end, this new tensor is called the *Ricci* tensor (*Rij*).

Is said that Einstein tried his final equation with the latter curvature tensor, but it did not work as it gave contradictory results, then he modified it to create a new tensor called the Einstein's tensor **(G)** to substitute the Ricci's one**.** Finally, he expressed the mass-energy tensor by T and the constant was found to be k = 8***π*** (in natural units[1]). Finally, the field equation was:

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta} \tag{15}$$

With **G** as:

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \tag{16}$$

This equation allows to determine the deformation of a field by T , also and the inverse. Note that both **T** and **G** have many components and in each case or solution one must treat them differently. Be also aware that here G is the Einstein's tensor, not the gravitational constant.

The next part will deal mainly the Schwarzschild solution and the Newton case.

1.3-SOME GR's SOLUTIONS

In this section we will discuss some of the solutions that General Relativity has. Mainly the Schwarzschild solution and the Newtonian case. This is the last part before the reaching the key point of this article, Mercury's orbit.

Now will be introduced the concept of giving a metric to a space-time, as it is crucial to calculate distances, trajectories and curvature. With the metric tensor can be solved the geodesic equation or can found the Riemann tensor and consequently these which are obtained with it. The metric tensor comes from the coordinate system, has a matrix structure and are diagonal, at least which are used here. This tensors, generally called *g*, are produced by the dot product between basis of vectors of a given coordinate axes:

$$\vec{e}_\alpha \cdot \vec{e}_\beta \equiv g_{\alpha\beta} \tag{16}$$

Examples of metric tensors are:

$$\delta_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad S_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta \end{pmatrix} \quad \cdots$$

[1] Natural Units are where the speed of light c and the gravitational constant G (among others) equal to 1.

the first one is the Newtonian metric, the second one is the Minkowski's one (as in SR) and the last one is the metric for spherical coordinates. To calculate intervals between points, i.e invariant distances, the metric is used as follows. Eq. 17 is called the line element:

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$$

$$(Euclidean\ case)$$
$$ds^2 = g_{00}dt^2 + g_{11}dx^2 + g_{22}dy^2 + g_{33}dz^2 = 1dt^2 + 1dx^2 + 1dy^2 + 1dz^2 \tag{17}$$

where as always *d* indicates a differential in any any axis, the *g* with sub-index indicates the component of *g* and ds^2 as the total distance between events (the interval).

The Minkowski metric above is a solution of Einstein's equations in the absence of sources, i.e., a vacuum solution of GR with no curvature. It is in fact the unique vacuum in the absence of a cosmological constant. As we already discussed, SR is simply physics in rigid Minkowski space-time, while in the presence of massive objects the space-time becomes curved and the metric deviates from the Minkowski metric, giving rise to general relativistic effects.

It is important to treat another metric that is found by solving the Einstein's equations for a special and concrete case but also a very common one. This metric is called the Schwarzschild metric, and is the metric which expresses the space-time outwards a gravitational source. It's usually derived from the spherical metric and is found considering that the only source of gravity is a spherical object, in consequence, that the gravitational field has spherical symmetry, also that the whole system is static (in space), i.e invariant in time. That the metric should be based on the mass and radius, since for zero mass and infinite radius the metric must be reduced to Minkowski (i.e to the SR). The Schwarzschild solution, based on spherical spaces, can be expressed by the line element in Eq. 18 as:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) \cdot dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} \cdot dr^2 + r^2 d\Omega^2 \tag{18}$$

with:

$$d\Omega^2 = d\theta^2 + sin^2(\theta)d\phi^2 \tag{19}$$

In Eq. 18 appears *G* as the gravitational constant (do not to confuse it with the Einstein tensor), *M* the mass of the source, and *r* as the radial distance between the source and the coordinate system.

Terms in gold are the "added" to the Minkowski metric in a spherical space and are functions that are due to the curvature of space-time. This tool is very useful and allows to predict the behavior from the time and space under the effect of gravity. Since Christoffel symbols depend on the metric, now the geodesic equation can be solved for free falling objects.

We will see that the motion a particle that follows a geodesic can't be described accurately enough by the Newtonian equations.

We should consider that the object moves in with a non-relativistic speed, i.e that the time components of the velocity are much larger than space ones. Starting with the geodesic equation and using as the parameter, and also as the proper time of the free-falling object:

$$\frac{\partial V^{\alpha}}{\partial \tau} + \Gamma^{\alpha}_{\gamma\beta} V^{\gamma} V^{\beta} = 0 \tag{20}$$

If the components of the geodesic equation are mostly time-like, we can simplify Eq. 20, taking that $V^0 >> V^1$. It can be seen that the significant part of the Christoffel symbols comes from V^0. Because the objective is to represent how the body's radial velocity changes, can be written Eq. 21 according to this. As always (t = 0, r = 1 ...):

$$\frac{\partial V^{1}}{\partial \tau} + \Gamma^{1}_{00} V^{0} V^{0} = 0 \tag{21}$$

Special Relativity shows that $V^0 \simeq 1$ for small space velocities so the Eq. 21 simplifies to:

$$\frac{\partial V^{1}}{\partial \tau} = -\Gamma^{1}_{00} \tag{22}$$

Now it's just left to calculate the Christoffel symbol, on the right hand side of the Eq. 22. This is done from the metric, which in this case is the Schwarchild one:

$$\Gamma^{1}_{00} = -\frac{1}{2} g^{11} g_{00,1} \tag{23}$$

In Eq. 23 appears the comma notation (",") in the last term and indicates, in this case, a derivative with respect to x^1. It also appears g^{11} and that is only the inverse of g_{11}. If the components of the metric are replaced by the corresponding values we get:

$$\Gamma^{1}_{00} = -\frac{1}{2}\left(1 - \frac{2GM}{r}\right)\cdot\left(-\frac{2GM}{r^2}\right) = -\frac{1}{2}\left(-\frac{2GM}{r^2} + \frac{(2GM)^2}{r^3}\right) \tag{24}$$

The last term of Eq. 24 can be neglected in a first order. This approach would give Eq. 25:

$$\Gamma^{1}_{00} = -\frac{1}{2}\left(-\frac{2GM}{r^2} + \dots\right) \simeq \frac{GM}{r^2} \tag{25}$$

Replacing the Christoffel symbol in Eq. 22:

$$\frac{dV^{1}}{d\tau} = a \simeq -\frac{GM}{r^2} \tag{26}$$

where "a" is the acceleration. Eq. 26 is predicted by Newtonian theory, and is the well-known Universal Gravitation law, but in this case derived from GR. This shows that the speed at which objects fall does not depend on their mass, as expected. It is important to stress here that it required to neglect several terms to find exactly the same equation as in the Newtonian case. For higher speeds or for a source with a huge mass, the equations would differ because the terms that have been neglected in Eq. 25 would be significant.

Now lets trade another of the predictions that GR has: under the effect of a gravitational field the time of a non-inertial observer (who is near a static source) becomes slower than for another inertial observer free of gravity, i.e, at nearly infinite distance *r* from the source.

Let’s calculate how significant this is using only the Schwarzschild and Minkowski metrics (the second can be obtained as a limit of the first when the radius *r* tends to infinity). Imagine two events separated only temporarily on earth, for example the birth of a person and his death. It is known that an average person lives about 82 years. The goal now is to calculate how much time the person living on earth would win compared to someone who has gone free of gravity far away. The two line elements are written below:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \qquad (26)$$

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) \cdot dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} \cdot dr^2 + r^2 d\Omega^2 \qquad (27)$$

In Eq. 26, 27 the red part is zero because the interval is only temporal. For both the proper time (which seems to pass for them) *is -ds≡dτ*. For the observer at infinity *dτ is* equal to the coordinate time *dt,* but for the observer on earth this is not true. For the second observer the proportion between *dτ* and dt is given by the following equation:

$$dt'^2 = \left(1 - \frac{2M}{r}\right) \cdot dt^2 \qquad (28)$$

As *dt* is the time coordinate, which coincides with the proper time of the inertial observer, we can use Eq. 28 as the relation between the time of the observers. The part in brackets of Eq. 28 has no units. Kilograms should be converted into meters as natural units are used here:

$$M_\oplus = 5.9736 \cdot 10^{24} (kg) \cdot \frac{Gc^{-2}(m)}{1(kg)} = 4,435 \cdot 10^{-3} (m)$$

$$R_\oplus = 6.371 \cdot 10^6 (m)$$

$$\mathrm{M}_\oplus \equiv \mathrm{Earth's\ mass}, \quad \mathrm{R}_\oplus = \mathrm{Earth's\ radius}$$

Now we can calculate the constant of proportionality between the two observers’ times*:*

$$d\tau = \left(\sqrt{1 - \frac{2[4.435 \cdot 10^{-3}]}{6.371 \cdot 10^6}}\right) \cdot dt$$

$$d\tau = \sqrt{0.999999998608} \cdot dt$$

Using these equations we find the time difference between an observer that spends 82 years at radius *r =earth radius* and another observer at r = *infinity*. This difference is only about 1.8 seconds for a lifetime on earth! This gives an idea of how small is the curvature caused by our planet and the relativistic effects it creates.

4. MERCURY’S PERIHELION PRECESSION.

During more than two centuries Newton's theory of gravity and Kepler's laws were perfectly adequate to calculate and describe the orbits of the planets in the solar system. An example of the effectiveness of Newton's theory is the discovery of Neptune in 1846. Uranus had periods of acceleration and others of slowdown relative to a Newtonian orbit along its path around the Sun, this suggested the existence of a nearby planet that was producing these irregularities to Uranus. Two scientists, Le Verrier and John Couch Adams, did a laborious task to calculate the exact position of the planet at a certain time using only classical mechanics. The objective was to see through the most powerful telescopes to look for this new planet. Actually, Le Verrier's prediction was very accurate, and with only 1º of error the planet was found where it was expected. Afterwards, Le Verrier tried with another planet that was moving strangely, it was Mercury. He had observed that the orbit was rotating on itself, i.e., the perihelion[1] was rotating by some fraction of a degree at each period, see Fig. 3. He thought of the existence of a new planet, as with he did with Neptune. First he had to consider the perturbations caused by all other planets like Venus, Earth, etc... So from the initial 5600 arcsec/century (1 arcsec= 1/3600 degrees) there were only 43 arcsec/century unexplained.

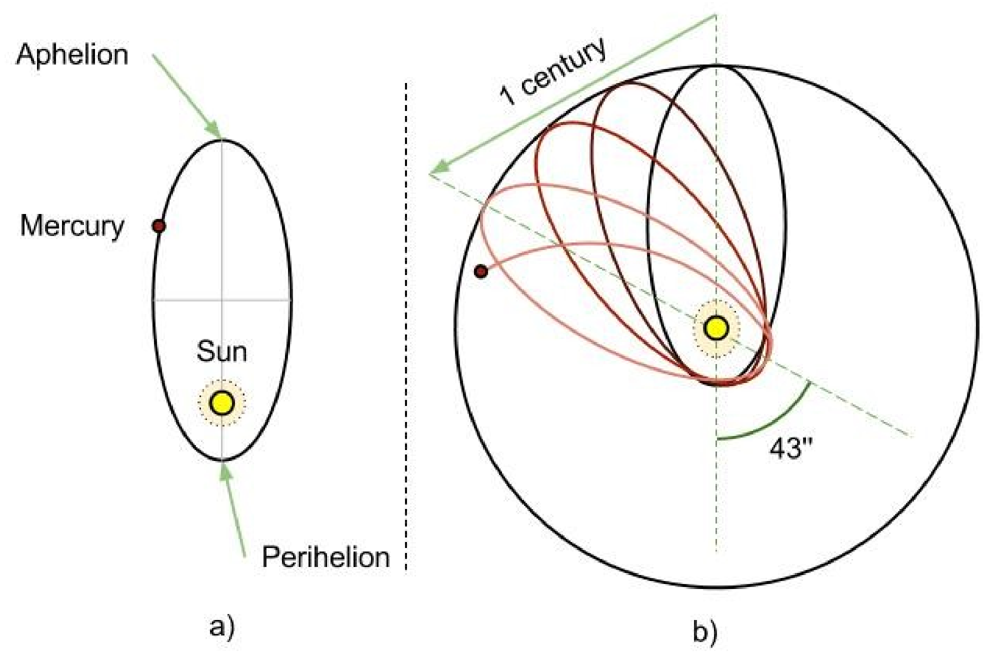

Fig. 3- In a) Mercury describes a "Newtonian" closed orbit, which means that after a period the planet returns to the same point. Therefore in b) Mercury follows a relativistic orbit where the perihelion shifts during each period. This movement is only about 43"/century.

[1] The perihelion of an orbit is the nearest point from the planet to the star during its trajectory. Also there is the aphelion which is the opposite of the perihelion as it is the point where the distance is the largest.

Le Verrier made some calculations to determine the characteristics of the hypothetical planet which he named Vulcan. After extensive research no one could find it so the hypothesis was refused. Another possibility that could explain the precession was that the sun was not really a perfect sphere (a little bit like the earth), i.e, a globe with the equatorial diameter wider than the polar.

Scientists as R. Dicke, M. Goldenberg, and H.Hill, among others, investigated the Sun's shape for more than 45 years but finally there wasn't any agreement on the final result due to the large experimental error they had. The newest data comes from the NASA's X and Gamma-ray telescope (RHESSI) in November of 2008. It measured 8.01 ± 0.14 miliarcsec, which corresponds to 6km of difference between the polar and equatorial diameter. The oblation that NASA measured would not be enough to ~~consider it the~~ cause ~~of~~ the precession of Mercury's orbit or even to have any significant contribution to it. In 1915 A. Einstein was finishing his new theory of gravity, and he found that his theory explained the planetary movements in a slightly different way. In fact, he predicted the precession of the orbits. For Mercury the prediction was, surprisingly, about 42.98" / century, the amount unexplained by Newtonian theory. The calculation comes from deriving the Schwarzschild metric and the geodesic equation to find the radius of any orbit as a function of the angle, momentum, mass, etc. To derive the formula one has to keep in mind the condition that the modulus of *p* (momentum 4-vector) is always equal to -m^2 regardless of the observer. In fact, the modulus comes from the line element (Eq. 18) since a 4-vector is some interval in 4 spacetime dimensions. The objective now is to calculate the exact amount of precession Mercury from what we have seen during the paper.

For simplicity are defined the following constants, *E* and *L*:

$$E \equiv -\frac{p_0}{m} \qquad (29)$$

$$L \equiv \frac{p_\phi}{m} \qquad (30)$$

It should be taken into account that $p^0 = g^{00} p_0$ (same for index *Phi).* If the condition is that the module of *p* is constant, it's true that:

$$-m^2 = g_{tt} \cdot (p^t)^2 + g_{rr} \cdot (p^r)^2 + g_{\phi\phi} \cdot (p^\phi)^2 + g_{\theta\theta} \cdot (p^\theta)^2 \qquad (31)$$

with $g_{\alpha\beta} \equiv 1/g^{\alpha\beta}$ and $p^\theta = 0$ as any orbit can be described in one plane, so we can choose *Theta=0*. Taking into account the Eq. 29 and 30 can be combined with Eq. 31 to derive Eq. 32 that determines the variation of the radius relative to the parameter *Tau* or proper time along the orbit / trajectory:

$$\left(\frac{dr}{d\tau}\right)^2 = E^2 - \left(1 - \frac{2M}{r}\right) \cdot \left(1 + \frac{L^2}{r^2}\right) \qquad (32)$$

To calculate the precession, it is interesting to study the variation of the radius depending on the angle, this is done by dividing two known derivatives:

$$\left(\frac{dr}{d\phi}\right) = \left(\frac{dr}{d\tau}\right) / \left(\frac{d\phi}{d\tau}\right) = \frac{\sqrt{E^2 - \left(1 - \frac{2M}{r}\right) \cdot \left(1 + \frac{L^2}{r^2}\right)}}{1/r^2 \cdot L} \tag{33}$$

As can be seen in Eq. 33 the blue fraction comes from the definition of L and its relation with the angular velocity (the blue term on left of the second equals sign).

Interest now focuses on solving this differential equation for *r* in terms of *φ*. Two approximate solutions of interest can be found, describing accurately the Newtonian and the relativistic cases. In the Newtonian case, terms with $1/r^3$ are neglected and in the relativistic solution, terms with $(1/r - M/L^2)^3$ are ignored (these are negligible considering that the planet's orbit is nearly Newtonian). We can guess the general form of the final equation knowing that it should be a cyclical process as the radius, after a period, is again the same (although it was an elliptical path). It's not surprising that the function is a kind of a sinusoidal equation:

$$r = (r' + A\cos(k\phi + B))^{-1} \tag{34}$$

The constants *r', A* and *k* are combinations of *L, E, M* and *B* is an initial angle. The most interesting part here is the constant *k*, written in green in Eq. 34. The radius oscillates around *r'* each time that the term inside the cosine is equal to *2π*. In the Newtonian approximation the constant k is equal to 1 and so, as the orbit goes around *2π*, the planet returns at the same point. In the relativistic case, however, $k<1$ and, since radius is the same once $k\varphi=2\pi$, the planet must revolve by more than *2π* to return to the same radius. This is exactly what we mean by precession of the planetary orbit.

The constant *k* has the following value (which is found from Eq. 33):

$$k = \sqrt{\left(1 - \frac{6M^2}{L^2}\right)} \simeq \left(1 + \frac{3M^2}{L^2}\right)^{-1} \tag{35}$$

The angle needed to complete the period of *r* , taking the approximate value of *k* in Eq. 35, is:

$$\phi = 2\pi/k = 2\pi\left(1 + \frac{3M^2}{L^2}\right) \tag{36}$$

So the angle difference with the full turn (2π) is the *delta Phi* as the value of the precession:

$$\Delta\phi = 2\pi\left(1 + \frac{3M^2}{L^2}\right) - 2\pi = 6\pi\frac{M^2}{L^2} \tag{37}$$

We can see that the precession of the perihelion is a prediction for any orbit in general, the study of Mercury it's because it has a higher deviation due to its near orbit to the Sun.

Now we can proceed to calculate the exact precession for Mercury based on Eq. 37. The first step is to simplify the above equation by introducing $L^2=Mr$, a value found for nearly Newtonian and circular orbits like Mercury. We then can reduce Eq. 37 and replace it with the values of the average radius and mass of the Sun to find out the exact angle (in natural units):

$$\Delta\phi \simeq 6\pi \frac{M_{\odot}}{r}$$

$$M_{\odot} = 1.47\ (km); \quad r = 5.55 \cdot 10^7\ (km);$$

$$\Delta\phi \simeq 6\pi \frac{1.47}{5.55 \cdot 10^7} = 4.99 \cdot 10^{-7} rad$$

(38)

This last result is the extra angle for each revolution. To see how much small is this value you can think about the angle we would measure for an ant 6km away. Therefore this deviation is usually expressed in units of arcsec/century:

$$\Delta\phi = 4.99 \cdot 10^{-7} \frac{rad}{orbit} \cdot \frac{180^{\circ}}{\pi\ rad} \cdot \frac{3600''}{1^{\circ}} \cdot \frac{1\ orbit}{0.24\ years} \cdot \frac{100\ years}{1\ century} = 43''/century$$

The 43'' are what was left to explain of the total observed 5600''. They are those degrees that *Le Verrier* could not attribute to any planet, the deviation that Dicke and Goldenberg could not explain by the Sun's oblation, those that the Newton's theory of Universal Gravitation did not predict. The 43'' that only GR could give the answer to.

5. CONCLUSIONS

Finally, we have demonstrated that Mercury, and in fact any other planet, should have a precession of its perihelion when orbiting around the Sun. It was necessary to introduce the theory of General Relativity from the beginning with basic concepts to be able to follow and understand the process of deriving the formula for the precession. We first treated the theory's postulates, the principle of equivalence and differences and similarities with Newtonian gravity. Then we dealt with one of the key concepts of the theory, the geodesic equation, the parallel transport and curvature of space-time. Later we approached the curvature tensors and finally the Einstein's field equations and their implications. Also the Schwarzschild solution as a key to find out the precession's equation and the Newtonian case. Finally we derived the perihelion's shift formula, and we could solve it for Mercury.

A conclusion of what has been traded throughout the paper, is that the theory of general relativity introduced a completely different way of seeing gravity: the gravitational forces disappeared and were replaced by space-time warping. The new theory introduced a lot of new concepts as black holes, gravitational radiation, gravitational time dilation and the redshifting of light, the cosmological constant, etc. The precession prediction gave some credibility to Einstein's theory in its early ages as it was one of the first pieces of observational evidence supporting the theory. The prediction was only made by GR because this theory of gravity is not just an alternative to the Newtonian one. Einstein's theory includes the classical planetary motion as a first approximation when weak fields are studied, so, it’s logical that when looked at high precision the differences between the two theories emerge. GR is a theory that included special relativity, but unlike it, GR was not found from any test or experimental evidence. One could say that scientific method was applied somewhat at the inverse... first the theory was published and then scientists looked for predictions to subject it to experimental tests.

6. REFERENCES

1. SCHUTZ, B. (1990). *A first course in general relativity*. Cambridge, GB: University Press.

2. GEROCH, R. (1985). *La Relatividad general (de la A a la B). Madrid, ES*: Alianza Editorial.
Original title: *General Relativity from A to B.*

3. WILL, C. (1989). *¿Tenía razón Einstein?*. Barcelona, ES: Editorial Gedisa S.A.
Original title: *¿Was Einstein Right?*.

Web references:

1. http://teoria-de-la-relatividad.blogspot.com.es/2009/03/indice.html [from 01-08-12 to 05-01-13]

2. http://forum.lawebdefisica.com/activity.php [from 01-08-12 to 05-01-13]

3. http://en.wikipedia.org/wiki/Mercury_(planet) [04-01-13]

4. http://www.wolframalpha.com/ [from 01-08-12 to 05-01-13]

7. ACKNOWLEDGEMENTS

I wanted to thank all those who has supported me during the nearly six month that has last the learning and writing process. First thank my tutors, Dr. Anastasios Avgoustidis and Dr. Carlos Martins, for their help always I needed, for the hours they spend on me and finally for trust me when I said that I wanted to write an article about General Relativity nearly without previous knowledge on the topic. Thanks to the organizers of the project “Joves i Ciència” (Youngs and Science) of the Catalunya Caixa Foundation, and especially to Eva Calvés for admitting me in the program and giving this great chance to youth. Also I want to thank my tutors from the high school *s’Agulla* (Blanes) who helped me during the selection process. Also to the UPC (Technical University of Catalonia) who lend me the needed books during the whole process. Thanks to my family who supported me every day and always encouraged me. I couldn’t stop thanking because there is a lot of people that has made this paper possible, for all them, thank you very much!