Strange Attractors in dissipative dynamical systems

Aleix Ciudad

1 Introduction to Chaos

The word *chaos* comes from the hellenic $\chi\acute{\alpha}o\varsigma$ (opening) when it was used to talk about the origin of the known world before Gods came and turned it into Cosmos ($\kappa\acute{o}\sigma\mu o\varsigma$) The chaotic is related with the unpredictable, unexpected, disordered, erratic... but if we want to get in deeper water, a little more conciseness is required.
What scientists call chaos is only a special part of all our incapacity to predict the future (main feature of the physics in the last centuries). We can separate this incompetence in three aspects:stochasticity,spatiality and chaos. It's impossible to perform an exact prediction when at least one of the previous characteristic is present. In the first case, because of completely random noise. In the second, the infinity of degrees of freedom makes the problem infinitely hard. But, surprisingly, the chaotical behavior is deterministic.
"So what's the problem?" one could ask. "All is governed by the equations". This yields to the fundamental property of chaos: *quasi* similar systems in an initial time will evolve to completely different situations, that is, two close trajectories in the phase space become exponentially separated with time. It means that any initial difference between them will grow becoming crucial. That's the reason why we call this chaos. Because our limited precision in any measure will be nothing compared with the growing error.

2 Introduction to Attractors

Nonlinear equations, the ones describing the most complex phenomena, are very difficult and usually impossible to deal with. That was the reason why Henry Poincaré, in 1892, introduced a new point of view, more qualitative but not useless at all. It consisted to analyze the behavior of trajectories in the phase space, leading to concepts like bifurcations, stability, fixed points, attractors...
It's not easy to define what an attractor is. Roughly speaking we can say that is a set to which all neighboring trajectories converge. But a more precise definition can be formulated (Strogatz 1994). We call a closed set A attractor if next three properties hold.

1. A has to be an invariant set, that is, any trajectory $\vec{x}(t)$ starting in A stays in A for all time.

2. A attracts an open set of initial conditions: There is an open set U containing A such that if $\vec{x}(0) \in U$,then the distance from $\vec{x}(t)$ to A tends to zero as $t \to \infty$. The largest U holding this is called the *basin of attraction* of A.

3. A is undecomposable: there is no subset of A satisfying 1 and 2.

Any single stable fixed point or limit cycle is an example of attractor.

3 Dissipative dynamical systems

A dissipative system is a system where Liouville's theorem doesn't hold anymore and consequently any volume of the phase space will decrease with time. Let V be the volume enclosing a surface S and let

$$\dot{\vec{x}} = \vec{F}(\vec{x}), \qquad \vec{x} = (x_1, \ldots x_d) \tag{1}$$

be the dynamical equations governing the flow. Then, the volume V will evolve

$$\frac{dV}{dt} = \int_V d^d x (\sum_{i=1}^{d} \frac{\partial F_i}{\partial x_i}) \tag{2}$$

Thus, we define a dissipative system as the one with $\frac{dV}{dt} < 0$.

4 Definition of a strange attractor

A strange attractor has the two following properties:

1. It is an attractor in the sense of the definition of the previous section.

2. The reason to be *strange* comes from the fact that is very sensitive on the initial conditions. In other words: despite the volume contraction, two close initial points become exponentially separated for very long times.

How it is possible? How can we maintain volume contraction and chaotic behaviour without contradiction? In figure 1 we can see that the volume shrinks in some directions and streteches in others, while a refolding takes place to keep the volume evolution. This process is what produce a chaotic motion in the attractor.

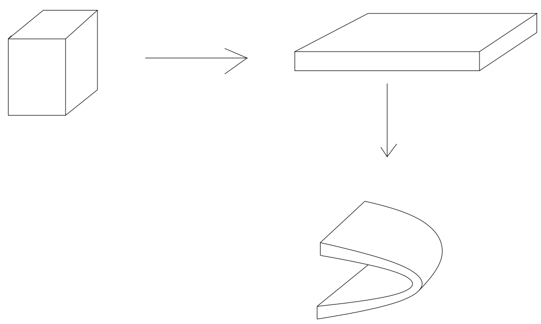

Figure 1: Volume evolution of the strange attractor

5 An example: The Lorenz Attractor

Lorenz, a meteoroligist of the XX century, proposed a model in which three coupled, first order, nonlinear differential equations led to complete chaotic trajectories.

$$\dot{X} = -\sigma X + \sigma Y \tag{3}$$

$$\dot{Y} = -XZ + rX - Y \tag{4}$$

$$\dot{Z} = XY - bZ \tag{5}$$

In figure 2 we can see a simulation of the model for $r = 28, \sigma = 10, b = \frac{8}{3}$. If we plot one of the coordinates versus time we will note the irregular

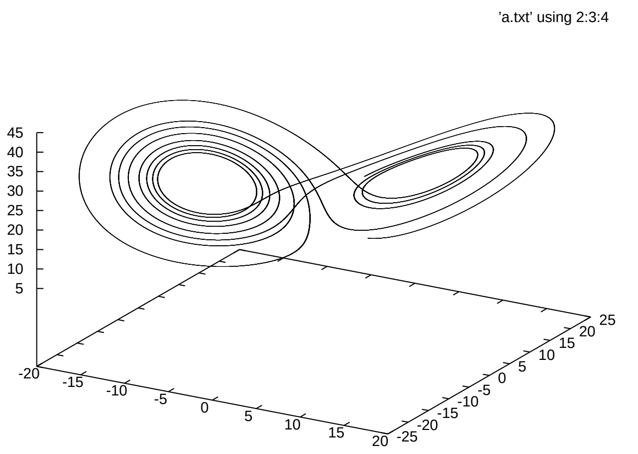

Figure 2: Lorenz attractor

oscillations. (Figure 3) We can also calculate the Liapunov exponent for the attractor.(Figure 4) Approximately, $\lambda = 4 \cdot 10^{-4}$.

6 The Kolmogorov Entropy

The Entropy introduced by Komogorov (1959) is the best characterization of any chotic motion. It's related with the thermodinamic entropy as well as with Liapunov exponents. Let's start defining it by considering a d dimensional trajectory $\vec{x}(t)$ of a dynamical system. We'll consider the d space divided in boxes l^d. Then, if we measure the system every time τ, the probability of $\vec{x}(t=0)$ being in box i_0, $\vec{x}(t=\tau)$ in box i_1, $\ldots$, and $\vec{x}(t=n\tau)$ in i_n will be $P_{i_0 \ldots i_n}$. Making use of the Shannon definition of entropy we get

$$K_n = -\sum_{i_0 \ldots i_n} P_{i_0 \ldots i_n} log(P_{i_0 \ldots i_n}) \tag{6}$$

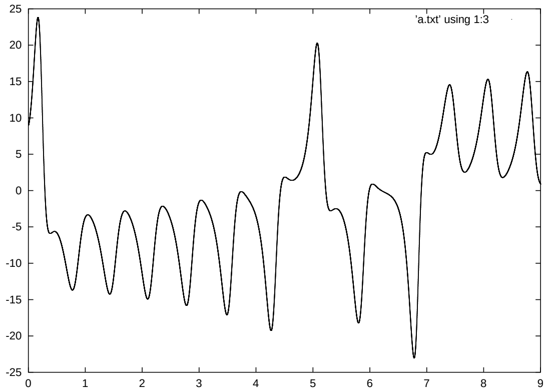

Figure 3: Aperiodic motion

$K_{n+1} - K_n$ is the loss of information from time $n\tau$ to time $(n+1)\tau$. So the Kolmogorov Entropy is defined as the average rate of loss of information, that is,

$$K = \lim_{\tau \to 0} \lim_{l \to 0} \lim_{N \to \infty} \frac{1}{N\tau} \sum_{n=0}^{N-1} (K_{n+1} - K_n) \tag{7}$$

In a one-dimensional map, K is just the (positive) Liapunov exponent λ. In higher dimensions is just the average of positive Liapunov exponents:

$$K = \int d^d x \rho(\vec{x}) \sum_i \lambda_i^+(\vec{x}) \tag{8}$$

7 Extraction information of the attractor from a signal

First of all, is useful to define the *generalized dimension* D_q. If we divide the attractor in boxes of linear dimension l and p_i is the probability that the attractor visits box i, then averaging powers q of the probabilities,

$$D_q = -\lim_{l \to 0} \frac{1}{q-1} | \frac{1}{log(l)} | log(\sum_i p_i^q) \tag{9}$$

This quantity is no more than a correlation between different points of the attractor, because $\sum_i p_i^q$ is, for $q > 1$, the total probability of having q points in one box. Sometimes we want to study the properties of an attractor but we can't measure all components of the trajectory at the same time (evident for an infinite dimensional time). Then, is useful to deal with the time series of one component in order to reconstruct some properties of the whole attractor (Takens 1981). If we have a d-dimensional flow

$$\frac{d}{dt}\vec{x} = \vec{F}(\vec{x}) \tag{10}$$

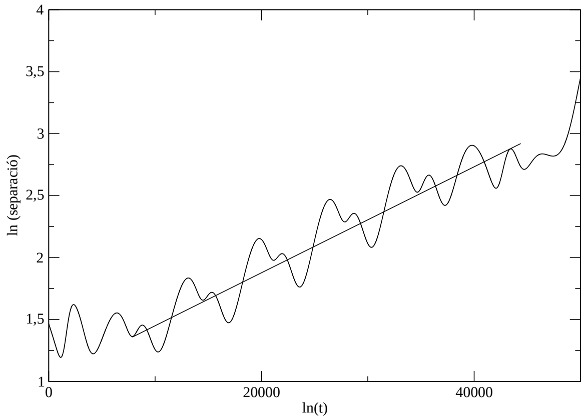

Figure 4: Liapunov exponent

then it can be proved (Takens 1981) that the vector

$$\vec{\xi}(t) = (x_j(t), x_j(t+\tau), ..., x_j(t+(2d+1)\tau)) \tag{11}$$

provides a smooth aproximation of the flow. Let's check this tool with the Rössler attractor (1976):

$$\dot{x} = -z - y \tag{12}$$

$$\dot{y} = x + ay \tag{13}$$

$$\dot{z} = b + z(x - c) \tag{14}$$

The direct simulation is shown in figure 5. Moreover, in figure 6 we reconstruct the attractor with $\tau = 0.92$. What's the meaning of τ? If we choose it very small, $x(t)$ and $x(t+\tau)$ will be indistinguishable and the realtion between them will be linear. The most reasonable choice is to take τ as close to the autocorrelation time as we can. (See figure 7). Another interesting quantity is the correlation integral $C(l)$ defined like

$$C(l) = \lim_{N\to\infty} \frac{1}{N^2} \sum_{ij} \theta(l - |\vec{x}_i - \vec{x}_j|) \tag{15}$$

from which one can get the correlation dimension D_2 as the slope in figure 8, where for the Rössler attractor we obtain $D_2 = 1.05$. This correlation integral can also be used to distinguish between deterministic irregularities and external white noise. If we have an strange attractor in a space where there is white noise, each point of the attractor will be sorrounded by a cloud of points in a d sphere of radius l_0 (the intensity of the noise). If $l \gg l_0$, the

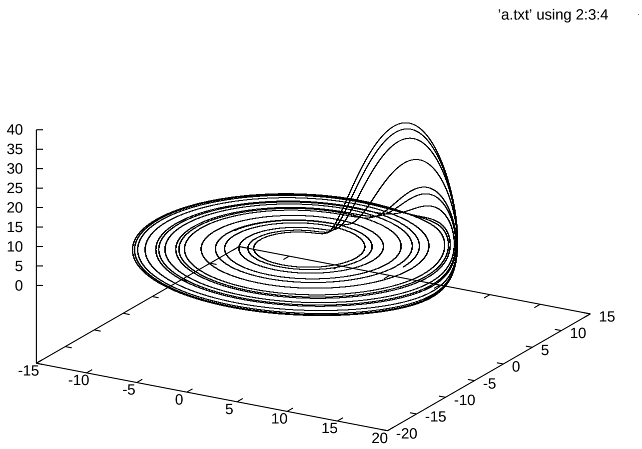

Figure 5: Rössler attractor

distance between points will not be affected very much, but if we increase the intensity of noise, then the distances between closer points will increase yielding to a lower value of $C(l)$. In figure 9 we can observe the effect of the noise for different values of its intensity.

8 The Julia and Mandelbrot sets

Let's consider the map

$$z_{n+1} = f_c(z) = z_n^2 + c \tag{16}$$

where $z_i, c \in \mathfrak{C}$. We say that a Julia Set J_c is the boundary of a basin of atraction of the map. In this case, the basin of atraction of $z* = \infty$ forms a Julia set J_c of $f_c(z)$. Julia (1981) and Fatou(1919) demonstrated that J_c is connected if and only if the following limit doesn't hold

$$\lim_{n\to\infty} f_c^n(0) \to \infty \tag{17}$$

The, one can ask: Which values of c make J_c connected? So the answer is: There is a set M for which the corresponding Julia set is connected. We say M is the Mandelbrot Set. In figure 10 we can see the general and popular aspect of M.

9 Bibliografy

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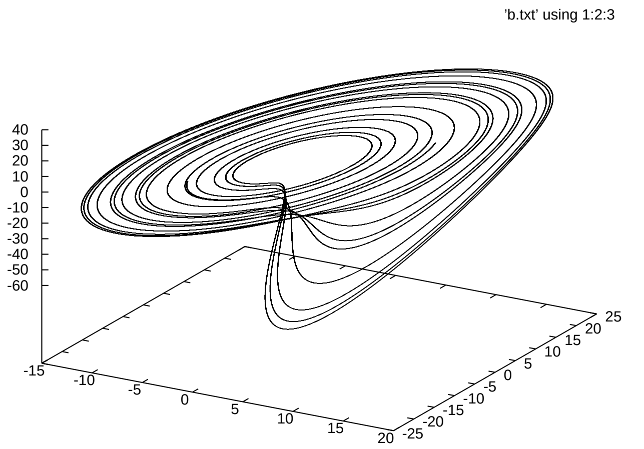

Figure 6: Reconstructed Rössler attractor

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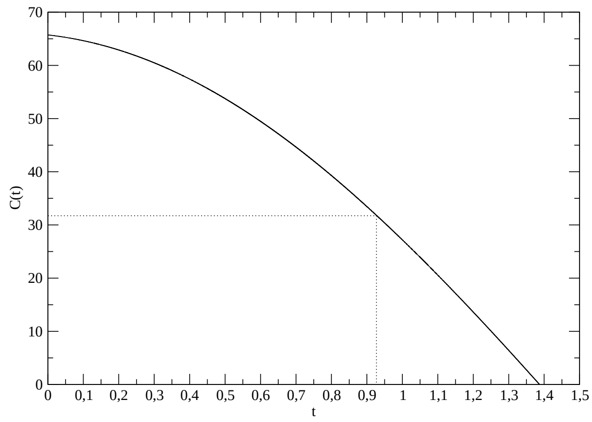

Figure 7: Time autocorrelation

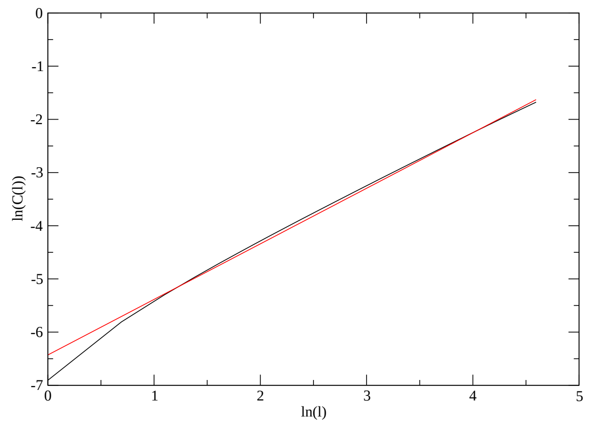

Figure 8: Correlation dimension

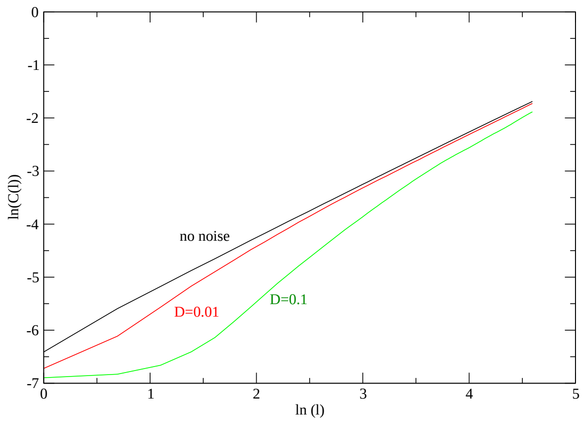

Figure 9: White noise/correlation dimension

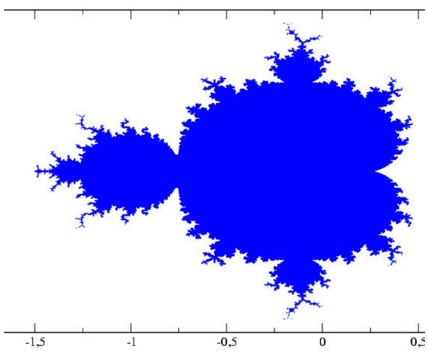

Figure 10: Mandelbrot Set